9.4 Application of Fourier Series

$$mx'' + cx' + kx = F(t)$$

- m ← mass
- c ← damping constant
- k ← spring constant

Solution: $x(t) = x_c(t) + x_p(t)$

- $x_c(t)$ ← complementary (homogeneous part)
- $x_p(t)$ ← particular (due to $F(t)$)

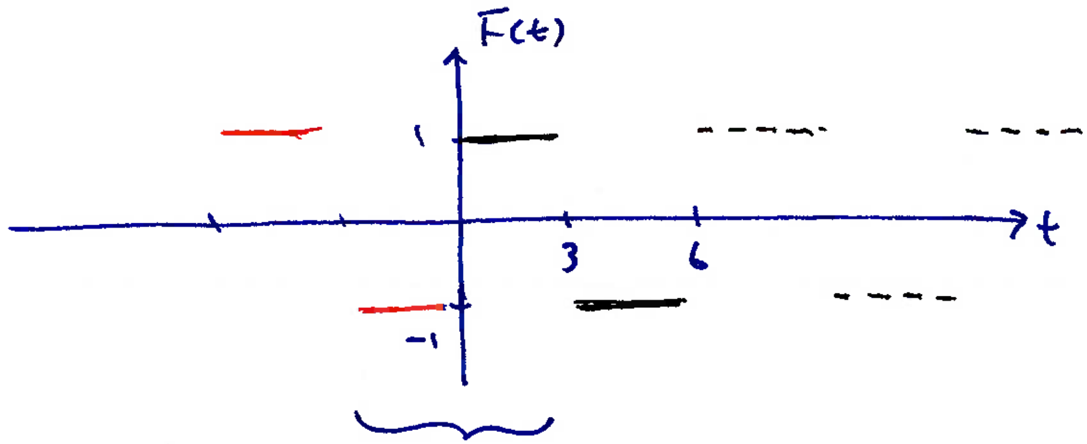
if $F(t) = F_0 \sin(\omega t)$ if $\omega \neq \sqrt{\frac{k}{m}}$

$$x_p = A\cos(\omega t) + B\sin(\omega t)$$

if $F(t)$ is periodic but not necessary sine / cosine

→ expand into sine and/or cosine series

for example, $c=0$, $m=1$, $k=5$ $\quad F(t) = \begin{cases} 1 & 0<t<3 \\ -1 & 3<t<6 \end{cases}$ period 6



origin symmetry

$F(t)$ is a <u>sine series</u> $\left(a_n = 0,\ b_n = \frac{2}{L}\int_0^L F(t)\sin\left(\frac{n\pi t}{L}\right)dt\right)$

⋮

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi t}{3}\right)$$

$$= \frac{4}{\pi}\sin\left(\frac{\pi t}{3}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi t}{3}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi t}{3}\right) + \cdots$$

$x'' + 5x = F(t)$ ←

each term results in its own particular solution in the form of $A_n \cos\left(\frac{n\pi t}{3}\right) + B_n \sin\left(\frac{n\pi t}{3}\right)$

then the entire particular solution is

$$x_p = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{3}\right) + B_n \sin\left(\frac{n\pi t}{3}\right)$$

it must satisfy

$$x'' + 5x = F(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi t}{3}\right)$$

sub x_p into the eq. above

⋮

$A_n = 0$ for all n

$$B_n = \frac{18\left[1-(-1)^n\right]}{n\pi(45-n^2\pi^2)}$$

general solution:

$$x(t) = \underbrace{c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)}_{x_c} + \underbrace{\sum_{n=1}^{\infty} \frac{18\left[1-(-1)^n\right]}{n\pi(45-n^2\pi^2)} \sin\left(\frac{n\pi t}{3}\right)}_{x_p \text{ (steady-periodic solution)}}$$

$45 - n^2\pi^2 \neq 0$ for $n = 1, 2, 3, \ldots$

but if $n = 2$, it is close to zero

because the input frequency $\frac{n\pi}{3} = \frac{2\pi}{3} \approx 2.094$

is close to the natural freq. $\sqrt{\frac{k}{m}} = \sqrt{5} \approx 2.236$

(near resonance)

can be a problem, but we are lucky here because

$$B_n = \frac{18\left[1-(-1)^n\right]}{n\pi(45 - n^2\pi^2)}$$ is 0 if $n=2$ $\left(\left[1-(-1)^n\right] = 0 \right)$

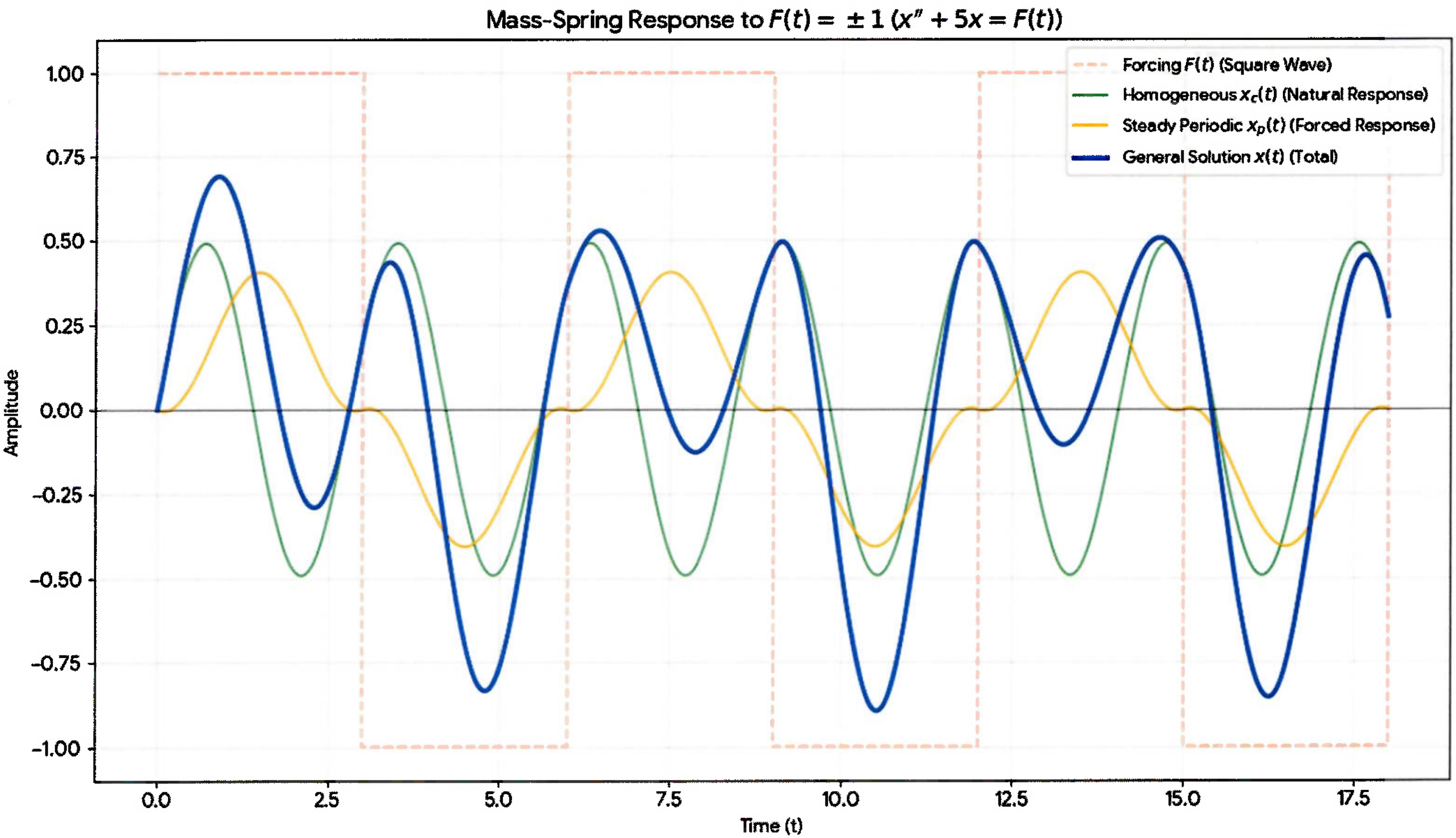
if we had $x'' + 5x = \begin{cases} 1 & 0 < t < 10 \\ -1 & 10 < t < 20 \end{cases}$ period 20

$\vdots$

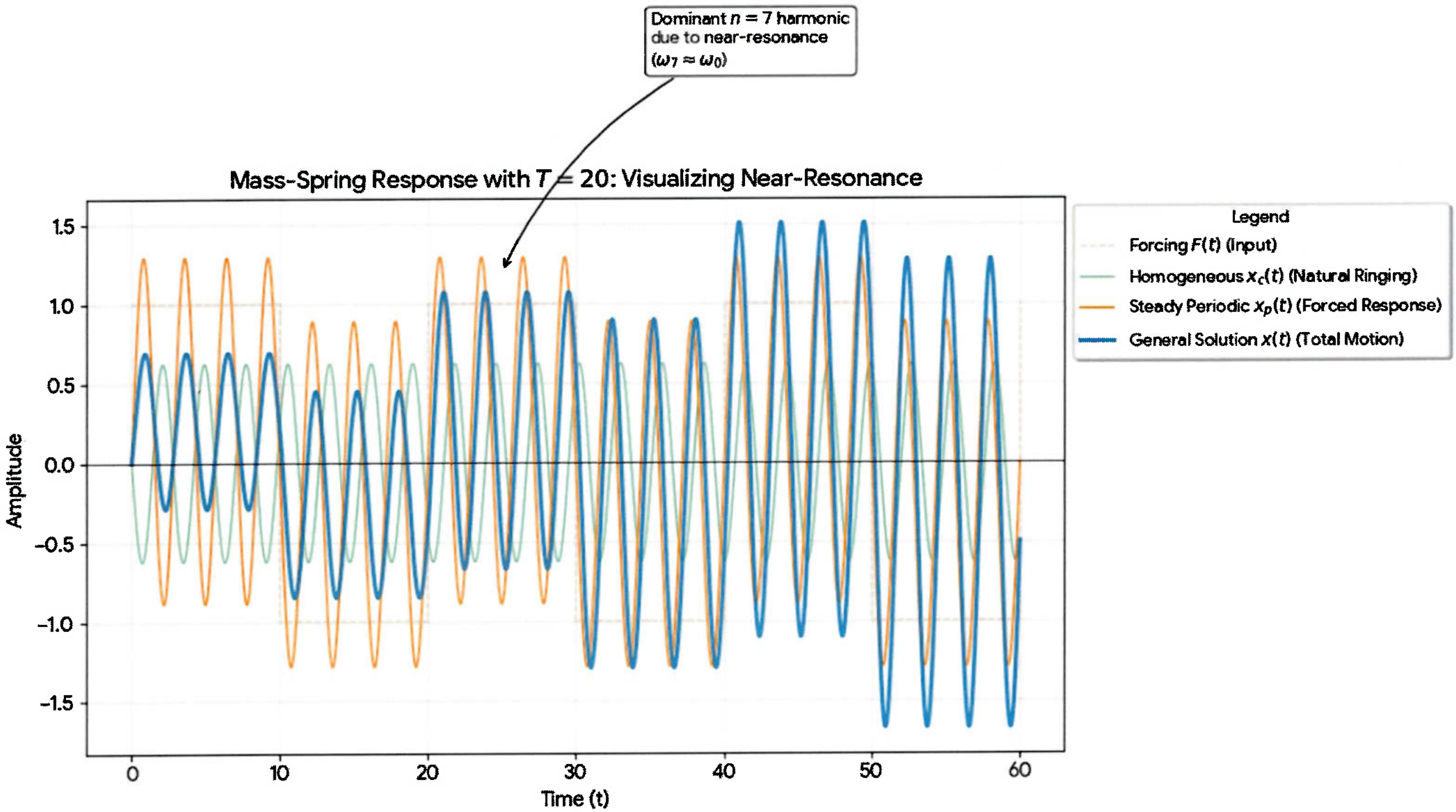
$$x_p = \sum_{n=1}^{\infty} \frac{200\left[1-(-1)^n\right]}{n\pi(500 - n^2\pi^2)} \sin\left(\frac{n\pi t}{10}\right)$$

$500 - n^2\pi^2$ is close to 0 if $n = 7$

$\left[1-(-1)^n\right]$ if $n=7$ is NOT zero $\rightarrow$ near resonance is visible



Mass-Spring Response to $F(t) = \pm 1$ $(x'' + 5x = F(t))$
Forcing $F(t)$ (Square Wave)
Homogeneous $x_c(t)$ (Natural Response)
Steady Periodic $x_p(t)$ (Forced Response)
General Solution $x(t)$ (Total)
Amplitude
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
Time (t)



Dominant $n = 7$ harmonic due to near-resonance ($\omega_7 \approx \omega_0$)
Mass-Spring Response with $T = 20$: Visualizing Near-Resonance
Amplitude
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
10
20
30
40
50
60
Time (t)
Legend
Forcing $F(t)$ (Input)
Homogeneous $x_c(t)$ (Natural Ringing)
Steady Periodic $x_p(t)$ (Forced Response)
General Solution $x(t)$ (Total Motion)

now look at $x'' + 9x = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$ period 2π

$\vdots$

$$x_p = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n(9-n^2)} \sin(nt)$$

$n=3$ makes $9-n^2=0$ exactly

part of $F(t)$ hits resonance freq. exactly (pure resonance)

the series solution x_p is not valid when $n=3$

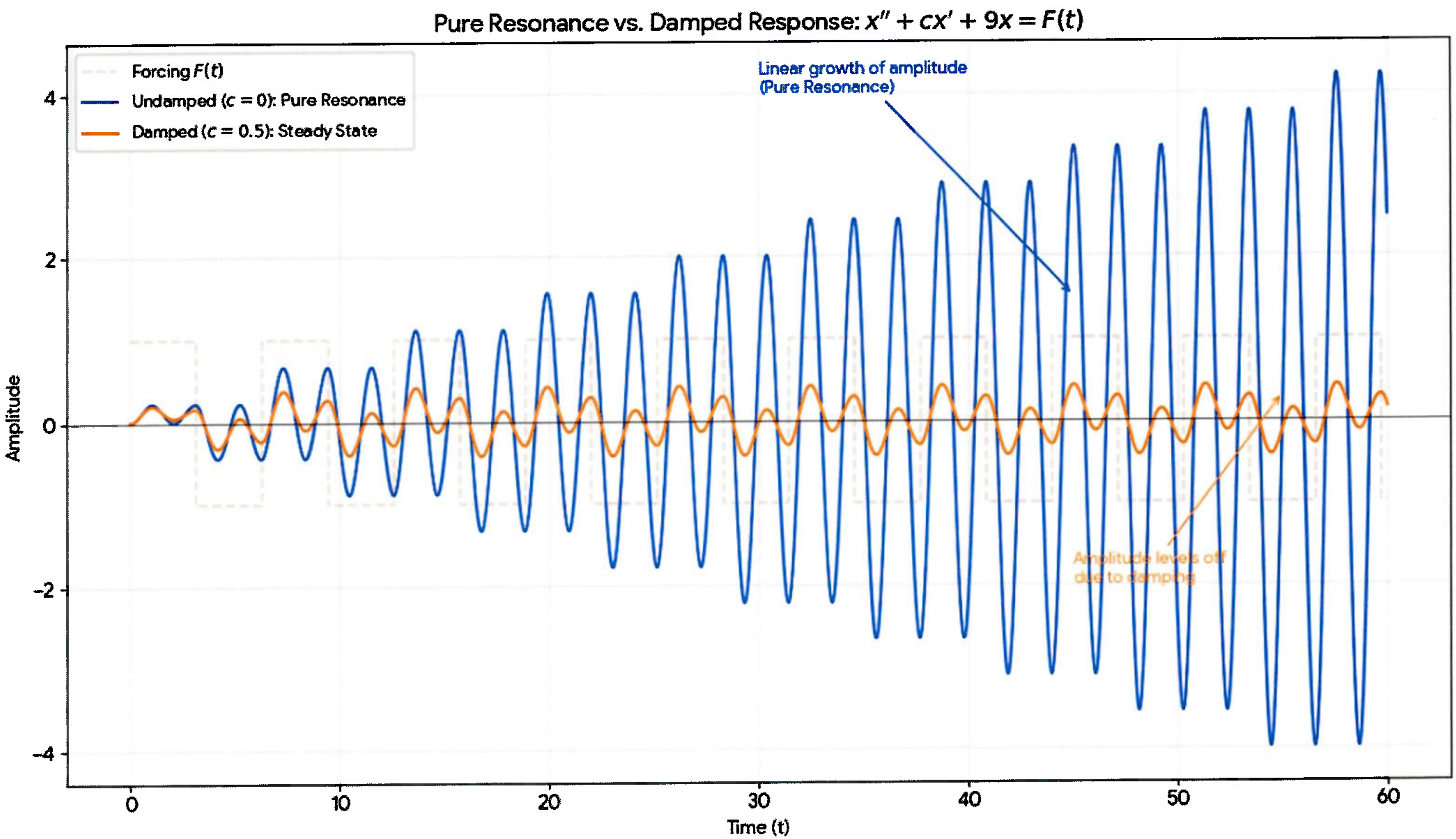
so, we need to handle that term separately

$$x_p = At\cos(3t) + Bt\sin(3t)$$

$\vdots$

$$A = -\frac{2}{3} \qquad B = 0$$

$$x_p = \underbrace{-\frac{2}{3}t\cos(3t)}_{n=3 \text{ only}} + \sum_{\substack{n=1 \\ n\neq 3}}^{\infty} \frac{2[1-(-1)^n]}{n(9-n^2)} \sin(nt)$$


Pure Resonance vs. Damped Response: $x'' + cx' + 9x = F(t)$
Forcing $F(t)$
Undamped ($c = 0$): Pure Resonance
Damped ($c = 0.5$): Steady State
Linear growth of amplitude (Pure Resonance)
Amplitude levels off due to damping
Amplitude
Time (t)
4
2
0
-2
−4
0
10
20
30
40
50
60

if $c \neq 0$, $mx'' + cx' + kx = F(t)$

$$F(t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi t}{L}\right)$$

$$x_p = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)$$

$\vdots$

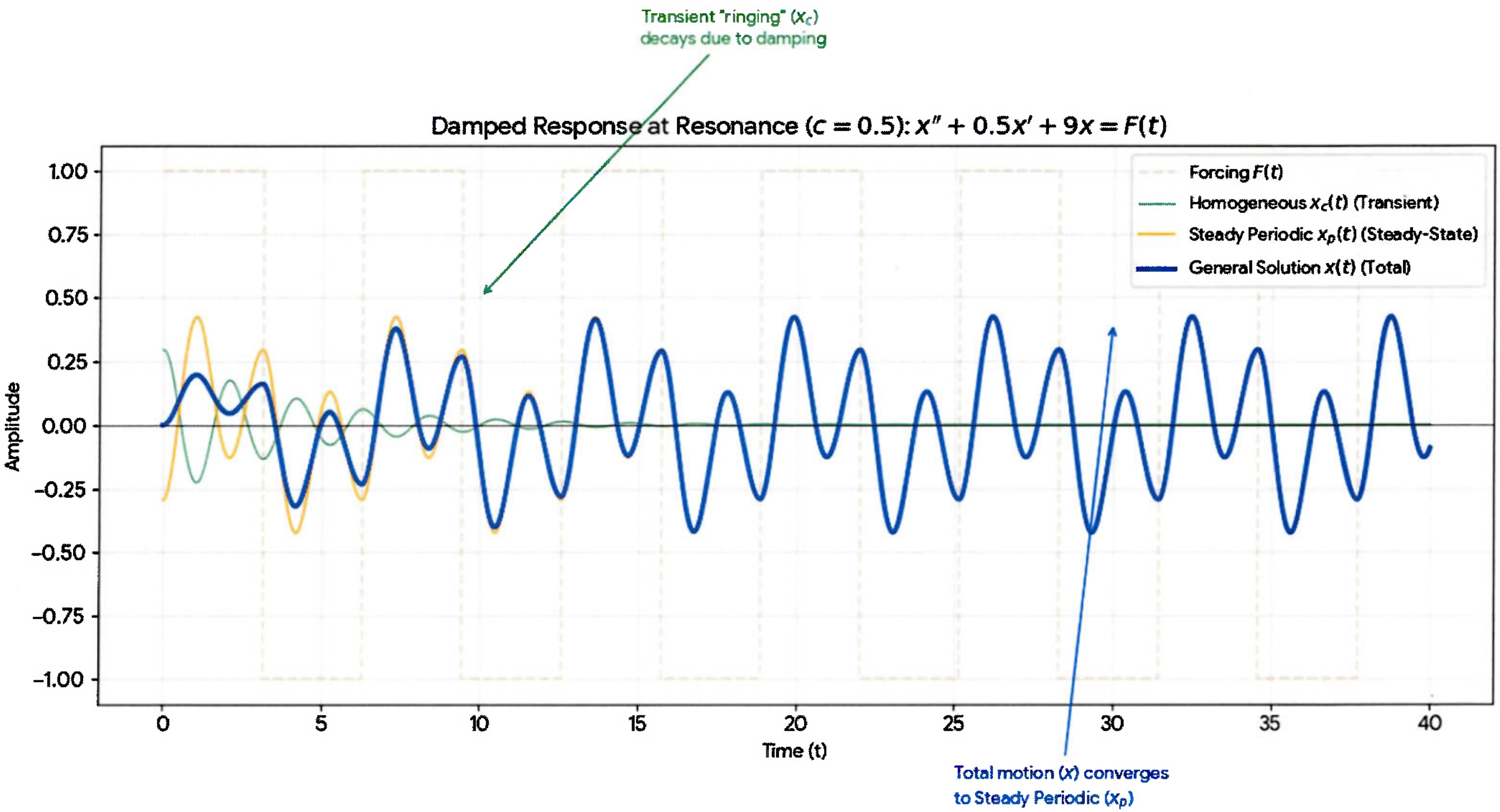
$$x_p = \sum_{n=1}^{\infty} \frac{F_n}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}} \sin\left(\frac{n\pi t}{L} - \underbrace{\tan^{-1}\left(\frac{c\omega_n}{k - m\omega_n^2}\right)}_{\text{phase shift}}\right)$$

$$\omega_n = \frac{n\pi}{L}$$

if $c \neq 0$ denom $\neq 0$

phase shift

particular solution lags behind complementary



Transient "ringing" (x_c)
decays due to damping
Damped Response at Resonance ($c = 0.5$): $x'' + 0.5x' + 9x = F(t)$
Forcing $F(t)$
Homogeneous $x_c(t)$ (Transient)
Steady Periodic $x_p(t)$ (Steady-State)
General Solution $x(t)$ (Total)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Amplitude
0
5
10
15
20
25
30
35
40
Time (t)
Total motion (x) converges
to Steady Periodic (x_p)